Limits and Derivatives Test

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December 12, 2021

Trigonometric Identities

Basic Identities

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \qquad \sec x = csc(\frac{\pi}{2} - x) \qquad \tan x = \cot\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \qquad \csc x = sec(\frac{\pi}{2} - x) \qquad \cot x = \tan\left(\frac{\pi}{2} - x\right)$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$

Angle Sum Identities

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

Double Angle Identities

$$\begin{aligned} \sin(2x) &= 2 \sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2\cos^2(x) - 1 \\ &= 1 - 2\sin^2(x) \\ \tan(2x) &= \frac{2\tan x}{1 - \tan^2 x} \end{aligned}$$

Log Rules

$\log_b x$ is defined for x only when $x > 0$. The $\log x$ function implies $\log_{10} x$ while then $\ln x$ function represents $\log_e x$.

1) Multiplication rule $\quad \log_b xy = \log_b x + log_b y$
2) Division rule $\quad \log_b \frac{x}{y} = \log_b x - log_b y$
3) Power rule $\quad \log_b x^n = n \log_b x$
4) Change of base rule $\quad \log_b x = \frac{\log_c x}{\log_c b}$
5) Exponent rule $\quad b^{\log_b x} = x$
6) Equality rule $\quad$ If $\log_b x = \log_b y$ then $x = y$
7) Identical base and operand rule $\quad \log_b b = 1$

Limits

A limit is defined by as follows:

$$\lim_{x \to c} f(x)$$

Indicating the number $f(x)$ tends to when x approaches c.
The above limit is two sided. To define a one-sided limit, we can use the following notations

$$\lim_{x \to c^-} f(x) \tag{1}$$

$$\lim_{x \to c^+} f(x) \tag{2}$$

The equation (1) refers to a limit when one approaches from the negative, or left-hand side of c. We note this as approaching from c^-. Equation (2) implies that we approach the limit from the positive or right-hand side, hence the c^+.

To evaluate a limit, we can simply plug in our value to the function. If we are able to obtain an answer, that is the result of the limit. However, if we are unable to determine an answer (perhaps due to a divide by zero), we can resort to methods such as simplifying the fraction or multiplying by the conjugate.

Example:

$$\begin{aligned}
&\lim_{x \to 2}(x^3 - 2x^2 + 3) \\
&= 2^3 - 2(2^2) + 3 \\
&= 3
\end{aligned}$$

This limit is simple to evaluate. Simply plugging in $x = 2$ yields 3 without any problems.

Example:

$$\lim_{x \to -3} \left(\frac{x^2 - 9}{x + 3} \right)$$

Here, we encounter some problems. If we try plugging in $x = -3$ we get:

$$\begin{aligned}
&\lim_{x \to -3} \left(\frac{x^2 - 9}{x + 3} \right) \\
&= \frac{(-3)^2 - 9}{-3 + 3} \\
&= \frac{9 - 9}{3 - 3} \\
&= \frac{0}{0}
\end{aligned}$$

Our result is undefined. In this case, we can resolve this issue by factoring our numerator and simplifying the fraction, eliminating the divide by zero problem.

$$\begin{aligned}
&\lim_{x \to -3} \left(\frac{x^2 - 9}{x + 3} \right) \\
&= \lim_{x \to -3} \left(\frac{(x - 3)(x + 3)}{x + 3} \right) \\
&= \lim_{x \to -3} (x - 3) \\
&= -3 - 3 \\
&= -6
\end{aligned}$$

Existence of Limits

The two-sided limit

$$\lim_{x \to c} f(x)$$

Only exists if $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ both exist, and

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

in which case

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x)$$

Example:

$$\lim_{x \to 1} \left(\frac{x^2(x-1)}{x-1} \right)$$

To determine whether this limit exists, we must evaluate both the left and right-hand limits individually.

Left-hand side:

$$\begin{aligned} &\lim_{x \to 1^-} \left(\frac{x^2(x-1)}{x-1} \right) \\ &= \lim_{x \to 1^-} x^2 \\ &= 1^2 \\ &= 1 \end{aligned}$$

Right-hand side:

$$\begin{aligned} &\lim_{x \to 1^+} \left(\frac{x^2(x-1)}{x-1} \right) \\ &= \lim_{x \to 1^+} x^2 \\ &= 1^2 \\ &= 1 \end{aligned}$$

Thus we can show that

$$\lim_{x \to 1^-} \left(\frac{x^2(x-1)}{x-1} \right) = \lim_{x \to 1^+} \left(\frac{x^2(x-1)}{x-1} \right) = 1$$

meaning that our limit exists, and it evaluates to a value of 1.

Intermediate Value Theorem

The Intermediate Value Theorem states that given a function $f(x)$ that is continuous between two values $x = a$ and $x = b$ with $a \leq b$, there must exist a value $x = c$ where $a \leq c \leq b$ and $f(x) = k$ where k is any number between $f(a)$ and $f(b)$.

The function will take on all the real numbers between $f(a)$ and $f(b)$ at some point from $x = a$ to $x = b$.

Squeeze Theorem

Given a function three functions f, g, and h that satisfy $g(x) \leq f(x) \leq h(x)$ for all x, if for some value c

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = k$$

for some constant k, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = k$$

Limit Rules

1) Addition/subtraction rule $\lim_{x\to c} f(x) \pm g(x) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
2) Constant rule $\lim_{x\to c} c \cdot f(x) = c \cdot \lim_{x\to c} f(x)$
3) Multiplication rule $\lim_{x\to c} f(x) \cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
4) Division rule $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$
5) Composition rule $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$

Important Limits

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to \infty} \frac{1}{x} = 0 \qquad \lim_{x\to -\infty} \frac{1}{x} = 0$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0} \frac{\sin(ax)}{x} = a$$

$$\lim_{x\to -\infty} \tan^{-1}(x) = -\frac{\pi}{2} \qquad \lim_{x\to \infty} \tan^{-1}(x) = \frac{\pi}{2}$$

$$\lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^x = e \qquad \lim_{x\to \infty} \left(1 + \frac{a}{x}\right)^{\frac{x}{a}} = e$$

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e \qquad \lim_{x\to 0} \left(1 + \frac{x}{a}\right)^{\frac{a}{x}} = e$$

Definition of Derivative

The derivative $f'(x)$ is defined for the function $f(x)$ at $x = c$ is defined by either of the following two limits

$$\lim_{\Delta h\to 0} \frac{f(c+\Delta h) - f(c)}{\Delta h}$$

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

Differentiability

A function $f(x)$ is differentiable at $x = c$ if it satisfies the following conditions:

1) $f(x)$ is continuous at $x = c$
2) $\lim_{x\to c} f(x)$ exists
3) $f'(c^-) = f'(c^+)$

Example:
To check the function $f(x) = x^2$ for differentiability at $x = 1$, we can note that x^2 is continuous for all real x, and that $\lim_{x\to 1^-} x^2 = \lim_{x\to 1^+} x^2 = 1$. Thus, given $f'(x) = 2x$:

$$\lim_{x\to 1^-} 2x = \lim_{x\to 1^+} 2x = 2$$

all our conditions for differentiability are satisfied, and x^2 is differentiable at $x = 1$.

Tangent Lines

The equation of the line tangent to the function $f(x)$ at $x = c$ is given as:

$$y - f(c) = f'(x)(x - c)$$

Normal Lines

The normal line of a function $f(x)$ at some $x = c$ is the same as the line perpendicular to the function at that point. The slope for the line is:

$$m = -\frac{1}{f'(c)}$$

Thus, the equation of the normal line is:

$$y - f(c) = -\frac{1}{f'(c)}(x - c)$$

Important Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(cx) = c$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(cx^n) =$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_b x) = \frac{1}{x \ln b}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(b^x) = b^x \ln b$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\csc x \cot x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1 - x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{-1} x\right) = -\frac{1}{1 + x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2 - 1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

Derivative Rules

1) Addition/subtraction rule $\quad \frac{\mathrm{d}}{\mathrm{d}x} f(x) \pm g(x) = f'(x) \pm g'(x)$

2) Product rule $\quad \frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

3) Quotient rule $\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

4) Chain rule $\quad \frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x)$

Parametric

Given parametric functions $x = f(t)$ and $y = g(t)$, we have the following formulae:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{g'(x)}{f'(x)}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}t}\Big/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$$

Polar

Given polar function $r(\theta)$

$$x = r(\theta)cos\theta$$
$$y = r(\theta)sin\theta$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{r'(\theta)cos\theta - r(\theta)\sin\theta}{r'(\theta)sin\theta + r(\theta)\cos\theta}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}\theta}\Big/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta}$$

L'hoptial's Rule

Given a limit:

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

If this limit evaluates to a result of the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ type, we have the following:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Derivative of Inverse

$$\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x} = \frac{1}{f'(f^{-1}(x))}$$

Mean Value Theorem

The MVT states that given a function $f(x)$ is continuous and differentiable at all points between two numbers a and b, there must exist a value c ($a \leq c \leq b$) such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

In other words, there must be be a value of c between a and b such that the line tangent to $f(c)$ has the same slope as the secant line formed by the points $x = a$ and $x = b$ on the function.