Integration Test (No Applications)

Bill Wang

February 4, 2022

1 Fundamental Theorem of Calculus

The fundamental theorem of calculus states that the derivative and integration operations are inverse functions of each other. Namely:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(x)\,dx = f(x)$$

From this, we can derive the following forumlae:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(x)\,dx = f(x)g'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{b} f(x)\,dx = -f(x)g'(x)$$

Example:
Simplify the following integral

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{2x} f(x)\,dx\\ &= f(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(2x)\\ &= f(x)\cdot 2\\ &= 2f(x)\end{aligned}$$

2 Basic Integration Formulae

$$\int a\,dx = ax + C$$
$$\int ax^n\,dx = \frac{a}{n+1}x^{n+1} + C$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x\tan x\,dx = \sec x + C$$

$$\int \csc x\cot x\,dx = -\csc x + C$$
$$\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$
$$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$
$$\int -\frac{1}{1+x^2}\,dx = \operatorname{arccot} x + C$$
$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + C$$
$$\int -\frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arccsc} x + C$$

3 Definite Integrals

A definite integral of an integral that has both an upper and a lower bound. In such a case, given $F(x)$ is the integration of the function $f(x)$, we have the following

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Note that in terms of integration, $F(b) - F(a)$ is commonly expressed as

$$F(x)|_a^b$$

4 Rules of Integrals

Addition/subtraction rule:

$$\int f(x) \pm g(x)\,dx \;= \int f(x)\,dx \;\pm \int g(x)\,dx$$

Constant rule:

$$\int af(x)\,dx \;= a\int f(x)\,dx$$

Definite integral bounds rule:

$$\int_a^b f(x)\,dx \;= -\int_b^a f(x)\,dx$$

Split integral rule:

$$\int_a^b f(x)\,dx \;= \int_a^c f(x)\,dx \;+ \int_c^b f(x)\,dx$$

5 Integration by Substitution

When integrating a nonbasic integral, we can sometimes try to integrate by substitution. To do this, we pick some part of the original integral and replace it with another variable, commonly u, and then take the derivative of u with respect to x. Then, we substitute u and $\mathrm{d}u$ into the function, and try to cancel out all the mentions of x and $\mathrm{d}x$. If we are successful, we can integrate.

Example:

$$\int \sin 3x\,dx$$

We do not have a basic formula for $\sin 3x$. We will try integration by substitution. Lets let $u = 3x$. Thus,

$$\mathrm{d}u = 3\,\mathrm{d}x$$
$$\mathrm{d}x = \frac{1}{3}\,\mathrm{d}u$$

Subbing in for $3x$ and $\mathrm{d}x$ into the original integral, we get

$$\int \sin u \cdot \frac{1}{3}\,du$$

By the constant rule, we can move $\frac{1}{3}$ out of the integral.

$$\frac{1}{3}\int \sin u\,du$$

This is a basic integration with respect to u, and so we solve this easily.

$$-\frac{1}{3}\cos u + C$$

Our last step is to substitute u back into our integral, and we get

$$-\frac{1}{3}\cos 3x + C$$

And this is indeed the answer.

Example:

$$\int e^{-x}\,dx$$

Note here the $-x$ makes this not a simple integration. We should use integration by substitution.

$$\begin{aligned}\text{Let } u &= -x \\ \mathrm{d}u &= -\,\mathrm{d}x \\ \mathrm{d}x &= -\,\mathrm{d}u\end{aligned}$$

Subbing in for u and du for $-x$ and ddx into the original integral, we have

$$\begin{aligned}&\int e^u \cdot -du \\ &= -\int e^u\,du \\ &= -e^u + C\end{aligned}$$

Replacing u with $-x$, our final solution is

$$-e^{-x} + C$$

6 Integration by Parts

When faced with a complex integral, we can try to break function to be integrated into two parts.One of the parts will be some function u, and the other part will be the derivative of some function v, and thus can be represented as dv. Then, the following formula holds true.

$$\int u\,dv \;= uv - \int v\,du$$

The proof of this will be left as an exercise to the reader (it stems from the product rule.)

We generally use integration by parts when our integral consists of the product of two functions. When choosing which part to differentiate and which part to integrate, we can follow the following order.:

1) Inverse Trigonometric Functions	$\arcsin x, \arctan x$, etc.
2) Logarithmic Functions	$\ln x, \log_2 x$, etc.
3) Polynomial Functions	x, x^3etc.
4) Trigonometric Functions	$\sin x, \csc x$, etc.
5) Exponential Functions	$e^x, 2^x$, etc.

We should differentiate the part that has the highest priority on this list. This can be memorized using the *very helpful* acronym of **ILPTE**.

Example:

$$\int x \sin x\,dx$$

Here, we can see that the integral is made of two parts multiplied together, namely the two functions of x and $\sin x$. The presence of these two functions is a strong indicator that we should integrate by parts.

Looking at our priority table above, we can see that the polynomial function x takes priority over the trigonometric function of $\sin x$. Thus, we should set $u = x$ and take the derivative of it, and we should therefore set $\mathrm{d}v = \sin x$, and integrate it.

Derivative of u

$$\begin{aligned}u &= x \\ \mathrm{d}u &= \mathrm{d}x\end{aligned}$$

Integration of $\mathrm{d}v$

$$\begin{aligned}\mathrm{d}v &= \sin x \,\mathrm{d}x \\ v &= -\cos x\end{aligned}$$

Note that for the intermediate integration step in the process of integrating by parts, we can omit the constant.

Recalling the formula for integration by parts, our original integral I can be written as

$$\begin{aligned}I &= \int x \sin x \, dx \\ &= uv - \int v \, du \\ &= x \cdot -\cos x - \int -\cos x \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x\end{aligned}$$

Appending the constant that comes with indefinite integrals, we get

$$-x \cos x + \sin x + C$$

Example:

$$\int \ln x \, dx$$

This integral is deceptive. It is one of the basic integral that were memorized, and nor does it appear to be a integral made of two separate functions. However, if we rearrange the function in the following way

$$\int 1 \cdot \ln x \, dx$$

We can see that our integral is indeed made of two parts! One of the parts is clearly the logarithmic function of $\ln x$, but the other (hidden) part is simply the number 1. It should be noted that 1 is technically a polynomial function of degree 0, since it may be written as $1 \cdot x^0$.

Obeying our order for differentiation when using integration by parts, we see we should differentiate $\ln x$ and integrate 1. Therefore:

$$\begin{aligned}u &= \ln x \\ \mathrm{d}u &= \frac{1}{x} \,\mathrm{d}x\end{aligned}$$

And

$$\begin{aligned}\mathrm{d}v &= 1 \cdot \mathrm{d}x \\ v &= x\end{aligned}$$

Using integration by parts, we can see our final integral I is

$$\begin{aligned}I &= uv - \int v \, du \\ &= x \ln x - \int x \frac{1}{x} \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x\end{aligned}$$

Simplifying and adding the mandatory constant, we can see that this integral evaluates to

$$x(\ln x - 1) + C$$

6.1 Nested Integration by Parts

Sometimes, when integrating by parts, after we may end up with a $\int v\,du$ that is hard to integrate. In such cases, it may be necessary to apply integration by parts a second time on the resulting integral to compute the answer. To illustrate this case, let us return to a variant of a familiar integral.

$$\int x^2 \sin x\,dx$$

This integration looks very close to the integral $\int x \sin x\,dx$ that we integrated previously, except it has an extra power of x. Nevertheless, it is still made of a trigonometric part ($\sin x$) and a polynomial part (x^2), so let us try and integrate by parts.

$$\begin{aligned} u &= x^2 \\ \mathrm{d}u &= 2x\,\mathrm{d}x \\ \\ \mathrm{d}v &= \sin x\,\mathrm{d}x \\ v &= -\cos x \end{aligned}$$

Using the integration formula, we can find our integral I to be

$$\begin{aligned} I &= x^2 \cdot -\cos x - \int -\cos x \cdot 2x\,dx \\ &= -x^2 \cos x + 2\int x \cos x\,dx \end{aligned}$$

Here we see a problem. $\int x \cos x\,dx$ is not an integral we can solve on sight. However, it is not to worry. Let us apply integration by parts again to this integral

$$I_2 = \int x \cos x\,dx$$

Once again, we find u and $\mathrm{d}v$

$$\begin{aligned} u &= x \\ \mathrm{d}u &= \mathrm{d}x \\ \\ \mathrm{d}v &= \cos x\,\mathrm{d}x \\ v &= \sin x \end{aligned}$$

Thus

$$\begin{aligned} I_2 &= \int x \cos x\,dx \\ &= x \sin x - \int \sin x\,dx \\ &= x \sin x + \cos x \end{aligned}$$

Hooray! The integral has been resolved. Now, we just need to plug I_2 back into our original expression

$$\begin{aligned} I &= -x^2 \cos x + 2\int x \cos x\,dx \\ &= -x^2 \cos x + 2\left(x \sin x + \cos x\right) \end{aligned}$$

Adding the constant, we get our integral to be

$$-x^2 \cos x + 2\left(x \sin x + \cos x\right) + C$$

For this problem, we were required to integrate by parts multiple times before arriving at the answer.

6.2 Looping Integrals

Consider the integral

$$\int e^x \sin x \, dx$$

Clearly, this integral is made of the product of two separate functions, so we are inclined to use integration by parts. Going ahead:

$$\begin{aligned} u &= \sin x \\ \mathrm{d}u &= \cos x \, \mathrm{d}x \\ \\ \mathrm{d}v &= e^x \, \mathrm{d}x \\ v &= e^x \end{aligned}$$

$$\begin{aligned} \therefore I &= uv - \int v \, du \\ &= e^x \sin x - \int e^x \cos x \, dx \end{aligned}$$

Our integral has not resolved. This could be a case of nested integration by parts, so lets try once more:

$$\begin{aligned} u &= \cos x \\ \mathrm{d}u &= -\sin x \, \mathrm{d}x \\ \\ \mathrm{d}v &= e^x \, \mathrm{d}x \\ v &= e^x \end{aligned}$$

$$\begin{aligned} I_2 &= e^x \cos x - \int e^x \cdot -\sin x \, dx \\ &= e^x \cos x + \int e^x \sin x \, dx \end{aligned}$$

Plugging this back into I, we have

$$\begin{aligned} I &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \end{aligned}$$

It looks like we are getting nowhere. Our integral has not resolved yet, and we seem to be cycling between $\sin x$ and $\cos x$ within it. However, we can notice that our original integral was $I = \int e^x \sin x \, dx$, which means we can replace our result with

$$I = e^x \sin x - e^x \cos x - I$$

(We simply substitute the second occurrence of $\int e^x \sin x \, dx$ with I).

Performing algebra, we can find that

$$\begin{aligned} 2I &= e^x \sin x - e^x \cos x \\ I &= \frac{1}{2} \left(e^x \sin x - e^x \cos x \right) \\ &= \frac{1}{2} e^x \left(\sin x - \cos x \right) \end{aligned}$$

Thus, our integration finishes with

$$= \frac{1}{2} e^x \left(\sin x - \cos x \right) + C$$

7 Integration by Partial Fraction

Take the following integral

$$\int \frac{2x+3}{x^2+5x+6}\,dx$$

By inspection, we cannot integrate this very easily. It is also not easy to solve this integral by u-substitution, and it is not a product of functions, so integration by parts does not apply. What we would like to be able to do is reduce both the top and bottom by a power of x. Doing so would result in a fraction resembling $\frac{a}{x-b}$, which, by substituting $u = x - b$ turns out to be quite easy to integrate (calculation left as an exercise to the reader). However, the fraction is not reducible, and so it is not easy to see how to get rid of the power of x on top.

This is where partial fractions come in. Let us suppose try to rewrite our $\frac{2x+3}{x^2-x+6}$ as the sum of two separate fractions, $\frac{A}{M} + \frac{B}{N}$, where A and B are constants, and M and N are degree 1 polynomials of the form $x+$ some constant. Examples of M and N include things such as $x + 3$ and $5x - 2$.

Now comes the hard part, what values do we set for A and B, and what should M and N be? Well, M and N are easy enough. Factoring our denominator, we can represent it as $(x + 3)(x + 2)$. Therefore, let us set $M = x + 3$ and $N = x - 2$. Our sum then becomes

$$\frac{2x+3}{x^2+5x+6} = \frac{A}{x+3} + \frac{B}{x+2}$$

Now, how to find A and B? We can do so by taking the common denominator of $\frac{A}{x+3}$ and $\frac{B}{x+2}$ and actually adding them.

$$\begin{aligned}
&\frac{A}{x+3} + \frac{B}{x+2} \\
&= \frac{A(x+2)}{(x+3)(x+2)} + \frac{B(x+3)}{(x+3)(x+2)} \\
&= \frac{Ax+2A}{(x+3)(x+2)} + \frac{Bx+3B}{(x+3)(x+2)} \\
&= \frac{Ax+2A+Bx+3B}{(x+3)(x+2)} \\
&= \frac{Ax+Bx+2A+3B}{(x+3)(x+2)}
\end{aligned}$$

Tidying this up a bit, we see that this is equal to

$$\frac{(A+B)x+2A+3B}{x^2+5x+6}$$

And this satisfies the following equality

$$\frac{2x+3}{x^2+5x+6} = \frac{(A+B)x+2A+3B}{x^2+5x+6}$$

As the denominators are equal, we can compare the numerators:

$$2x + 3 = (A + B)x + 2A + 3B$$

Here, we begin the process of inspections. Specifically, we compare the coefficients on the different terms.

Lets look at the x term. Clearly, by the left side of the equation, the coefficient must be 2. Looking on the right side, we see that at the only place x appears, it is accompanied by a coefficient of $A + B$. Thus $A + B = 2$. Looking at the constant term, it should be equal to 3. On the right side, both $2A$ and $3B$ are constant terms (terms without an x). Thus, $2A + 3B = 3$. Now we have a system of equations

$$\begin{aligned}
A + B &= 2 \\
2A + 3B &= 3
\end{aligned}$$

Solving this system of equations (once again left as an exercise to the reader), we can find that $A = 3$ and $B = -1$. Therefore,

$$\frac{2x+3}{x^2+5x+6} = \frac{3}{x+3} + \frac{-1}{x+2}$$

Replacing this in our integral yields

$$I = \int \left(\frac{3}{x+3} + \frac{-1}{x+2} \right) dx$$

By the addition rule, we have

$$\begin{aligned} I &= \int \frac{3}{x+3}\, dx \; + \int \frac{-1}{x+2}\, dx \\ &= 3 \int \frac{1}{x+3}\, dx \; - \int \frac{1}{x+2}\, dx \end{aligned}$$

Let's integrate these two separately. First, we will consider $\int \frac{1}{x+3}\, dx$. This is a u-substitution integral. Let $u = x + 3$. Thus

$$\begin{aligned} u &= x + 3 \\ \mathrm{d}u &= \mathrm{d}x \end{aligned}$$

Replacing this we have

$$\begin{aligned} &\int \frac{1}{u}\, du \\ &= \ln |u| \\ &= \ln |x + 3| \end{aligned}$$

Moving on to $\int \frac{1}{x+2}\, dx$. This situation is very similar, and we once again substitute u into it, this time for $u = x + 2$. The rest of the calculations are almost identical

$$\begin{aligned} u &= x + 2 \\ \mathrm{d}u &= \mathrm{d}x \end{aligned}$$

$$\begin{aligned} I &= \int \frac{1}{u}\, du \\ &= \ln |u| \\ &= \ln |x + 2| \end{aligned}$$

Thus, our final integral is

$$\begin{aligned} &3 \int \frac{1}{x+3}\, dx \; - \int \frac{1}{x+2}\, dx \\ &= 3 \ln |x + 3| - \ln |x + 2| + C \end{aligned}$$

Note that integration by partial fraction works best when the denominator of a function is easily factorable. It does not work at all if the degree of the numerator is higher than the denominator. In that case, you should use long division.

8 Trigonometric Substitution

Trigonometric substitution is a method of solving integrals by substituting parts of the integral with trigonometric functions. This method required a good understanding and memorization of basic identities. Though this topic is covered in AP Calculus BC, it is not a large part of the exam, and safe to skip studying if you are low on time.

The identities of relevance are

$$\sin^2\theta + \cos^2\theta = 1 \qquad\qquad \tan^2\theta = \sec^2\theta - 1$$

Trigonometric substitution only applies in narrow cases. Specifically, they can only work when certain features are present in the integral. Notably:

Feature present in integral	What you should substitute x with	$\mathrm{d}x$ will equal
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$\mathrm{d}x = a \cos\theta \, \mathrm{d}\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$\mathrm{d}x = a \sec^2\theta \, \mathrm{d}\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$\mathrm{d}x = a \sec\theta \tan\theta \, \mathrm{d}\theta$

Usually, trigonometric substitution can only be used when you have one of the above cases with a square root.

Example: Evaluate and solve

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx$$

Note that in this case, we see that $\sqrt{4-x^2}$ satisfies $\sqrt{a^2-x^2}$ if we let a equal 2. Therefore, we substitute x with $a\sin\theta$ and get:

$$\int \frac{\sqrt{4-(2sin\theta)^2}}{(2\sin\theta)^2}\,dx$$

While we're at it, we should also substitute dx. Taking the derivative of x, we get

$$x = 2\sin\theta$$
$$\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$$

Therefore

$$I = \int \frac{\sqrt{4-(2sin\theta)^2}}{(2\sin\theta)^2}\cdot 2\cos\theta\,d\theta$$

If we expand the top and bottom, we get

$$\begin{aligned} I &= \int \frac{\sqrt{4-4\sin^2\theta}}{4\sin^2\theta}\cdot 2\cos\theta\,d\theta \\ &= \int \frac{\sqrt{4\left(1-\sin^2\theta\right)}}{2\sin^2\theta}\cdot\cos\theta\,d\theta \\ &= \int \frac{2\sqrt{1-\sin^2\theta}}{2\sin^2\theta}\cdot\cos\theta\,d\theta \\ &= \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}\cdot\cos\theta\,d\theta \end{aligned}$$

Using our identity on the numerator, we get

$$\begin{aligned} I &= \int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cdot\cos\theta\,d\theta \\ &= \int \frac{\cos\theta}{\sin^2\theta}\cdot\cos\theta\,d\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta \end{aligned}$$

Using the identity again, we can obtain

$$I = \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta$$

Which we can break into two fractions to integrate separately:

$$\begin{aligned} I &= \int \left(\frac{1}{\sin^2\theta} - \frac{\sin^2\theta}{\sin^2\theta}\right)d\theta \\ &= \int \frac{1}{\sin^2\theta} - \int \frac{\sin^2\theta}{\sin^2\theta}\,d\theta \\ &= \int \csc^2\theta - \int 1\,d\theta \end{aligned}$$

Both of these can be integrated, to get

$$-\cot\theta - \theta$$

Now comes the hard part – changing θ back to x. First, we perform some algebra

$$x = 2\sin\theta$$
$$\sin\theta = \frac{x}{2}$$

Recalling **SOH CAH TOA** we can see that if $\sin\theta = \frac{x}{2}$, then there is a right triangle with a hypotenuse of 2 and the opposite side of θ of x. If we let y be the third (adjacent) side, then by Pythagorean theorem

$$y^2 = 2^2 - x^2$$
$$y = \sqrt{4 - x^2}$$

Since $\cot\theta = \frac{1}{\tan\theta}$, we should find $\tan\theta$. Once again, we use **SOH CAH TOA** to find that $\tan\theta$ is opposite over adjacent, or $\frac{x}{y}$ in this case.

$$\tan\theta = \frac{x}{y} = \frac{x}{\sqrt{4-x^2}}$$

$$\therefore \cot\theta = \frac{\sqrt{4-x^2}}{x}$$

Now onto θ. If $\sin\theta = \frac{x}{2}$, then $\theta = \arcsin\frac{x}{2}$. Therefore, our final integral is

$$I = -\frac{\sqrt{4-x^2}}{x} - \arcsin\frac{x}{2} + C$$

9 Improper Integrals

Improper integrals are definite integrals bounds where one of the bounds is ∞ or $-\infty$. An example would be the following integral

$$\int_2^\infty \frac{1}{x^2}\,dx$$

We will rewrite this integral with a limit. Note that the upper bound is now b.

$$\lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx$$

Next, we evaluate the integral as normal, keeping the limit

$$\lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx$$
$$= \lim_{b\to\infty} -\frac{1}{x}\Big|_2^b$$
$$= \lim_{b\to\infty} -\frac{1}{b} - -\frac{1}{2}$$
$$= \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{2}$$

Evaluating the limit, we get

$$0 + \frac{1}{2}$$
$$= \frac{1}{2}$$

Thus, an improper integral is treated as a regular definite integral, except you "plug in" ∞ instead of some value at the end. However, **you must use the limit notation** or you will be docked marks, even if there is no function reason to.

10 Initial Value

Given a function defined by an integral (eg. $f(x) = \int g(x)\,dx$) and a point on the same function, we can actually determine the explicit definition of the function by following the following steps

1) Integrate the function

2) Plug in the point

3) Solve for the integration constant C

Example:
Given $f(x)$ is defined by

$$\int \sqrt{x}\,dx$$

And $(1, 3)$ is a point on the graph of $f(x)$, find $f(x)$ in terms of x.

We first integrate our integral:

$$\begin{aligned}
&\int \sqrt{x}\,dx \\
&= \int x^{\frac{1}{2}}\,dx \\
&= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \\
&= \frac{2}{3}x^{\frac{3}{2}} + C
\end{aligned}$$

Now given that $(1, 3)$ is on the graph, we know that $x = 1$ and $f(x) = 3$. Therefore,

$$\begin{aligned}
3 &= \frac{2}{3}1^{\frac{3}{2}} + C \\
3 &= \frac{2}{3} + C \\
C &= \frac{7}{3}
\end{aligned}$$

Now, we can determine our function $f(x)$ to be

$$\begin{aligned}
f(x) &= \frac{2}{3}x^{\frac{3}{2}} + \frac{7}{3} \\
&= \frac{1}{3}\left(2x^{\frac{3}{2}} + 7\right)
\end{aligned}$$

10.1 An Alternate Method

Alternatively, we can determine the equation of a formula given some point $f(a) = b$, and a function defined by an integral

$$f(x) = \int g(x)\,dx$$

The final function can be expressed using the following definite integral

$$f(x) = f(a) + \int_a^x g(x)\,dx$$

This method is ideal if you prefer to evaluate definite integrals instead of solving for C. Recall that definite integrals do not require an integration constant.

Example:
Given $f(0) = 2$ and

$$f(x) = \int \cos x\,dx$$

By the alternate method:

$$\begin{aligned}
f(x) &= f(0) + \int_0^x \cos x \, dx \\
&= f(0) + (\sin x)\Big|_0^x \\
&= f(0) + (\sin x - \sin 0) \\
&= 2 + \sin x - 0 \\
&= \sin x + 2
\end{aligned}$$

The alternate method may sometimes be faster than the standard substitution method, particularly for physics problems where the integration constant C may be quite complex.

11 Approximations

Questions regarding integral approximation will generally be asked in the following form: Using some type of approximation, approximate the integral of a function $f(x)$ from a to b with n equal subdivisions.

No matter they type of subdivision, we must determine how "wide" each subdivision is. We generally call this value Δx. This value is given by

$$\Delta x = \frac{b-a}{n}$$

Now, the precise formula for calculating the approximation depends on the type of approximation.

11.1 Left Riemann Sum

The Left Riemann Sum treats each subdivision as a "rectangular box" who's height is determined by the y-value of the function at the leftmost boundary of the subdivision. It's formula is given by

$$S = f(a)\Delta x + f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \cdots + f(a+(n-1)\Delta x)\Delta x$$

Which can be simplified to

$$S = \Delta x \left(f(a) + f(a+\Delta x) + f(a+2\Delta x) + \cdots + f(a+(n-1)\Delta x)\right)$$

And can be written with sigma notation

$$\Delta x \cdot \sum_{i=0}^{n-1} f(a+i\Delta x)$$

11.2 Right Riemann Sum

The Right Riemann Sum also treats each subdivision as a "rectangular box", but the height is determined by the y-value of the function at the rightmost boundary of the subdivision. It's formula is given by

$$S = f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \cdots + f(a+(n-1)\Delta x)\Delta x + f(b)\Delta x$$

Which can be simplified to

$$S = \Delta x \left(f(a+\Delta x) + f(a+2\Delta x) + \cdots + f(a+(n-1)\Delta x) + f(b)\right)$$

And can be written with sigma notation

$$\Delta x \cdot \sum_{i=1}^{n} f(a+i\Delta x)$$

11.3 Midpoint Riemann Sum

Instead of the left and right boundaries, this time we use the y-value at the middle of the boundary. It's formula is more complicated.

$$S = \Delta x \left(f(a + \frac{\Delta x}{2}) + f(a + \frac{3\Delta x}{2}) + \cdots + f(a + \frac{(2n-1)\Delta x}{2}) \right)$$

Or it's sigma notation

$$\Delta x \cdot \sum_{i=1}^{n} f(a + \frac{(2i-1)\Delta x}{2})$$

11.4 Trapezoidal Approximation

Instead of using rectangular boxes, we can be more accurate if we use a trapezoid, where the y-values of the functions at both the left and right sides of the function mark the to different heights of a trapezoid. Using the formula for area of a trapezoid, we have

$$S = \Delta x \left(\frac{f(a) + f(a + \Delta x)}{2} + \frac{f(a + \Delta x) + f(a + 2\Delta x)}{2} + \cdots + \frac{f(a + (n-2)\Delta x) + f(a + (n-1)\Delta x)}{2} + \frac{f(a + (n-1)\Delta x) + f(b)}{2} \right)$$

Which can be simplified to

$$= \frac{\Delta x}{2} \left(f(a) + 2f(a + \Delta x) + 2f(a + 2\Delta x) + \cdots + 2f(a + (n-1)\Delta x) + f(b) \right)$$

The sigma notation for the trapezoidal approximation is

$$\frac{\Delta x}{2} \sum_{i=0}^{n-1} f(a + i\Delta x) + f(a + (i+1)\Delta x)$$

11.5 Estimation Accuracy

Refer to the following table for when an approximation will overestimate or underestimate

Approximation	Underestimate when	Overestimate when
Left Riemann Sum	$f(x)$ is increasing	$f(x)$ is decreasing
Right Riemann Sum	$f(x)$ is decreasing	$f(x)$ is increasing
Trapezoidal Sum	$f(x)$ is concave down	$f(x)$ is concave up

It is not trivial to determine whether Midpoint Riemann will over or underestimate, and it will not be tested. Of course, the relationships depicted in the table can be calculated on demand, simply by drawing model functions of all 4 combinations of slope and concavity and looking to see if the approximation is over or underestimating.

11.6 Approximating Approximations

We generally use Riemann or Trapezoidal sums to estimate an integral, but there also may be questions that use an integral to estimate one of these sums.

Example:
Estimate the value of

$$\frac{1}{10} \left(\sqrt{4} + \sqrt{4 + \frac{1}{10}} + \sqrt{4 + \frac{2}{10}} + \cdots + \sqrt{9 - \frac{1}{10}} \right)$$

The $\frac{1}{10}$ at the front and the fact that the function increases by that same value every term should tip us off that this is some kind of approximation. Looking at it, we can see that it is the approximation for $\sqrt{x}$. Looking at the bounds, we can see that it begins at 4 and ends at 9. Therefore, this sum can be approximated by the integral

$$\int_4^9 \sqrt{x}\, dx$$

Which evaluates out to be $\frac{38}{3}$. Thus, $S \approx \frac{38}{3}$. The approximation sign is very important, and not writing it will lose marks.

12 Cheesing Integrals

Take the integral

$$\int_{-3}^{3} \sqrt{9 - x^2}\, dx$$

You may be tempted to use substitution or trigonometric substitution for this problem. However that make take a long time. Instead, we can avoid doing any integration if we think back to math 12.

Recall that the formula for the equation of a circle is

$$x^2 + y^2 = r^2$$

Solving for y we get

$$y = \pm\sqrt{r^2 - x^2}$$

Now doesn't that look like what we have in our integral? Since we don't have the $\pm$, we only have half the circle. However, we can see that $r^2 = 9$, and thus our radius is 3. Since this circle is centred at $(0, 0)$, the integral from -3 to 3 is actually just finding the area of the semicircle. By geometry the area is $\frac{1}{2}\pi 3^2$ or $\frac{9}{2}\pi$.

This strategy is perfectly legal on the AP, so if you can find a normal geometry formula to use, by all means use it.

13 Difficult Integration

13.1 Integration of Tangent

Solve

$$\int \tan x\, dx$$

We will rewrite this as

$$\int \frac{\sin x}{\cos x}\, dx$$

Here, we will perform u-substitution. We will let $u = \cos x$. Taking the derivative, we find that

$$\begin{aligned} u &= \cos x \\ \mathrm{d}u &= -\sin x\, \mathrm{d}x \\ \mathrm{d}x &= -\frac{1}{\sin x} \cdot \mathrm{d}u \end{aligned}$$

Here, we will substitute $\mathrm{d}x$ back in

$$\int \frac{\sin x}{u} \cdot \frac{-1}{\sin x}\, du$$

The $sinx$ on the top and bottom cancel out, so we get

$$\begin{aligned} &\int \frac{-1}{u}\, du \\ &= \int -\frac{1}{u}\, du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \end{aligned}$$

The $\cot x$ case is similar and has been left as an exercise to the reader.

13.2 Integration of Secant

Evaluate the following integral:

$$\int \sec x \, dx$$

To do this integration, we will multiply $\sec x$ by $\frac{\sec x + \tan x}{\sec x + \tan x}$. Since the numerator and denominator, are equal, we are essentially multiplying by 1, which is a valid operation.

$$\begin{aligned} &\int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\ =& \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \end{aligned}$$

Now, we will use u-substitution. Previously, we generally substituted u for simple x, such as $3x$ or $\frac{1}{4}x$. However, there is nothing preventing us from setting u to be multiple functions at once. In this case, we will set u to be equal to $\sec x + \tan x$. Thus

$$\begin{aligned} u &= \sec x + \tan x \\ \mathrm{d}u &= \left(\sec^2 x + \sec x \tan x\right) \mathrm{d}x \\ \mathrm{d}x &= \frac{1}{\sec^2 x + \sec x \tan x} \cdot \mathrm{d}u \end{aligned}$$

To perform this derivative, you must recall the derivative addition rule.

Now that we have $\mathrm{d}x$ in terms of u, we can substitute into the integral

$$\begin{aligned} &\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\ =& \int \frac{\sec^2 x + \sec x \tan x}{u} \cdot \frac{1}{\sec^2 x + \sec x \tan x} \, du \end{aligned}$$

Here, we can see something very beautiful. It so happens that the $\sec^2 x + \sec x \tan x$ terms cancel out!. Therefore, we are left with

$$\begin{aligned} &\int \frac{1}{u} \, du \\ =& \ln |u| + C \end{aligned}$$

Subbing u back in, we get

$$\ln |\sec x + \tan x| + C$$

The $\csc x$ case is similar and has been left as an exercise to the reader.

13.3 Deceptive Integration by Parts

Sometimes, an integral that looks like one that should be solved by parts is not in fact a by parts question.

$$\int x e^{-x^2} \, dx$$

Now, this integral is indeed two separate functions multiplied together, but we cannot actually integrate this easily using the by parts method. Instead, we can solve this by u-substitution.

$$\begin{aligned} \text{Let } u &= -x^2 \\ \mathrm{d}u &= -2x \, \mathrm{d}x \\ \mathrm{d}x &= -\frac{1}{2x} \, \mathrm{d}u \end{aligned}$$

Subbing this back into the integral, we get:

$$\int xe^u \cdot -\frac{1}{2x}\, du$$

And in this case, the x terms cancel out, and we are left with

$$\begin{aligned}
&\int -\frac{1}{2}e^u\, du \\
&= -\frac{1}{2}\int e^u\, du \\
&= -\frac{1}{2}e^u + C \\
&= -\frac{1}{2}e^{-x^2} + C
\end{aligned}$$

14 Average Value

The average value of a function $f(x)$ over the interval $a \leq x \leq b$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\, dx$$

This formula must be memorized.

Example:
Determine the average value of $f(x) = x^2$ over the interval $[1, 3]$.

$$\begin{aligned}
\text{Average Value} &= \frac{1}{3-1}\int_1^3 x^2\, dx \\
V_{\text{avg}} &= \frac{1}{2}\left(\frac{1}{3}x^3\right)\Bigg|_1^3 \\
&= \frac{1}{2}\left(\frac{1}{3}\cdot 3^3 - \frac{1}{3}\cdot 1^3\right) \\
&= \frac{1}{2}\left(9 - \frac{1}{3}\right) \\
&= \frac{13}{3}
\end{aligned}$$

Thus the average value of x^2 over from 1 to 3 is $\frac{26}{3}$.

15 Other Notes

Definite integrals never have a $+C$ at the end. If you add one, your answer will be deducted points.

The area of a circle is πr^2 and the area of a trapezoid is $\frac{1}{2}(a+b)\cdot h$

$1 - \frac{\pi}{4}$ is **not** $\frac{3\pi}{4}$.

On integration multiple choice, it is sometimes faster to take the derivative of every single answer and see which matches than to actually do the integration.

For improper integrals, just rewriting the integral with the limit is one mark on the free-response questions, regardless if you manage to progress further or not.