Derivatives (no applications)

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February 3, 2022

1 Normal lines

The normal line of a function $f(x)$ at some $x = c$ is the same as the line perpendicular to the function at that point. The slope for the line is:

$$m = -\frac{1}{f'(c)}$$

Thus, the equation of the normal line is:

$$y - f(c) = -\frac{1}{f'(c)}(x - c)$$

2 Definition of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

3 Basic derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[cx] = c$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[cx^n] = cnx^{n-1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_b x] = \frac{1}{x \ln b}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[b^x] = b^x \ln b$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cot x] = -\csc^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec x \tan x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\csc x] = -\csc x \cot x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1} x] = -\frac{1}{\sqrt{1 - x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1} x] = \frac{1}{1 + x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cot^{-1} x] = -\frac{1}{1 + x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2 - 1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

4 Finding tangent lines

The equation of the line tangent to the function $f(x)$ at $x = c$ is given as:

$$y - f(c) = f'(x)(x - c)$$

5 Addition, product, and quotient rules

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = \frac{\mathrm{d}f(x)}{\mathrm{d}x} \pm \frac{\mathrm{d}g(x)}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

6 Chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$$

7 Derivative of inverse

$$\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x} = \frac{1}{f'(f^{-1}(x))}$$

8 Parametric

Given parametric functions $x = f(t)$ and $y = g(t)$, we have the following formulae:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{g'(x)}{f'(x)}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}t}\Big/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$$

9 Polar

Given polar function $r(\theta)$

$$x = r(\theta)cos\theta$$
$$y = r(\theta)sin\theta$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{r'(\theta)cos\theta - r(\theta)\sin\theta}{r'(\theta)sin\theta + r(\theta)\cos\theta}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}\theta}\Big/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta}$$

10 L'hoptial's Rule

Given a limit:

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

If this limit evaluates to a result of the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ type, we have the following:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

11 Mean Value Theorem

The MVT states that given a function $f(x)$ is continuous and differentiable at all points between two numbers a and b, there must exist a value c ($a \leq c \leq b$) such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

In other words, there must be be a value of c between a and b such that the line tangent to $f(c)$ has the same slope as the secant line formed by the points $x = a$ and $x = b$ on the function.