Applications of Integration

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1 Area

Perhaps the most basic use of integration is to find the area under a curve. Here, we can just use normal integration. However, we note that we usually consider the "area" of a function to be strictly positive, even if it may lie under the x-axis. Therefore the formula for finding area under a curve is:

$$\int_a^b |f(x)|\, dx$$

Example:
Find the area bounded by the curve $f(x) = x^2$, the lines $x = 0$ and $x = 3$, and the x-axis.

Here, we just integrate $f(x)$ from 0 to 3. It is important to realize that x^2 is strictly positive for all x, and thus we can safely ignore the absolute value.

$$\begin{aligned}
&\int_0^3 f(x)\, dx \\
&= \int_0^3 x^2\, dx \\
&= \frac{1}{3}x^3\Big|_0^3 \\
&= \frac{1}{3}\left(3^3 - 0^3\right) \\
&= 9
\end{aligned}$$

Example:
Find the area bounded by the curve $f(x) = x^3 - 4x^2 - 11x + 30$, the lines $x = 0$ and $x = 5$, and the x-axis

Note that this function is not strictly positive for all x. Thus, the absolute value in the integration formula is relevant. To deal with this, we will split the integral into parts that are either completely positive or completely negative, adjusting the signs as necessary. To do this, we must first find the roots of $f(x)$. This calculation is somewhat trivial, and thus will not be shown. The roots calculate out to be $x = -3$, $x = 2$, and $x = 5$. Therefore, the integral becomes:

$$\begin{aligned}
&\int_0^5 f(x)\, dx \\
&= \left|\int_0^2 x^3 - 4x^2 - 11x + 30\, dx\right| + \left|\int_2^5 x^3 - 4x^2 - 11x + 30\, dx\right| \\
&= \left|\frac{x^4}{4} - \frac{4x^3}{3} + \frac{11x^2}{2} + 30x\right|_0^2 + \left|\frac{x^4}{4} - \frac{4x^3}{3} + \frac{11x^2}{2} + 30x\right|_2^5 \\
&= \left|\frac{94}{3}\right| + \left|-\frac{117}{4}\right| \\
&= \frac{94}{3} + \frac{117}{4} \\
&= \frac{727}{12}
\end{aligned}$$

1.1 Two functions

Sometimes, you will need to find the area bounded not by a function and the x-axis, but instead two different functions $f(x)$ and $g(x)$. In this case, our formula is much the same, except we use the distance between the two functions instead. The formula then becomes

$$A = \int_a^b |f(x) - g(x)|\, dx$$

As with our the x-axis case, we can get around the absolute value by determining the intervals in the integral where $f(x) > g(x)$ and when $f(x) < g(x)$. Then we can calculate the intervals separately and fix the signs after the fact to calculate the area.

Example:
Calculate the area bounded by the curves 2^x and x^2 as well as the y-axis ($x = 0$) and the line $x = 4$.

$$f(x) = 2^x$$
$$g(x) = x^2$$

We now need to find where these curves intersect:

$$2^x = x^2$$

By inspection, the solutions are $x = 2$ and $x = 4$. Therefore, we will evaluate these the intervals $(0, 2)$ and $(2, 4)$ separately, and add the positive areas afterwards.

$$A = \int_0^2 \left(2^x - x^2\right)\, dx \; + \int_2^4 \left(2^x - x^2\right)\, dx$$

Note that the integral $\int \left(2^x - x^2\right)\, dx \; = \frac{2^x}{\ln 2} - \frac{1}{3}x^3$. Using this, we evaluate the areas separately.

$0 \le x < 2$:

$$\begin{aligned} A &= \left(\frac{2^x}{\ln 2} - \frac{1}{3}x^3\right)\Big|_0^2 \\ &= \left(\frac{4}{\ln 2} - \frac{8}{3}\right) - \left(\frac{1}{\ln 2} - 0\right) \\ &= \frac{3}{\ln 2} - \frac{8}{3} > 0 \\ \therefore A &= \frac{3}{\ln 2} - \frac{8}{3} \end{aligned}$$

$2 \le x < 4$:

$$\begin{aligned} A &= \left(\frac{2^x}{\ln 2} - \frac{1}{3}x^3\right)\Big|_2^4 \\ &= \left(\frac{16}{\ln 2} - \frac{64}{3}\right) - \left(\frac{4}{\ln 2} - \frac{8}{3}\right) \\ &= \left(\frac{12}{\ln 2} - \frac{56}{3}\right) < 0 \\ \therefore A &= \frac{56}{3} - \frac{12}{\ln 2} \end{aligned}$$

Adding these two positive areas together we get

$$\begin{aligned} A &= \frac{3}{\ln 2} - \frac{8}{3} + \frac{56}{3} - \frac{12}{\ln 2} \\ &= \frac{48}{3} - \frac{9}{\ln 2} \\ &= 16 - \frac{9}{\ln 2} \end{aligned}$$

And this is our area.

1.2 Y-axis

Sometimes, instead of finding the area bounded by a function and the x-axis, we may be required to find the area of a function with respect to the y-axis. In this case, we will solve the function for y and then integrate with respect to y, using dy.

Example:
Find the area bounded by the curve x^3, the line $y = 0$, $y = 2$, and the y-axis.

Solving the function for x, we get:

$$y = x^3$$
$$x = \sqrt[3]{y}$$

Now we integrate this function with respect to y.

$$\begin{aligned}
&\int_0^2 \sqrt[3]{y}\, dy \\
&= \left.\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right|_0^2 \\
&= \left.\frac{3}{4}x^{\frac{4}{3}}\right|_0^2 \\
&= \frac{3}{4}\left(2^{\frac{4}{3}}\right) - 0 \\
&= \frac{3}{4}\sqrt[3]{16}
\end{aligned}$$

2 Average Value

The average value of a function from a to b can be given by the simple following integral:

$$\frac{1}{b-a}\int_a^b f(x)\, dx$$

Example:
Find the average value of $f(x) = \cos x$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

By the formula, we get:

$$\begin{aligned}
V_{\text{avg}} &= \frac{1}{\frac{\pi}{2} - (-\frac{\pi}{2})}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx \\
&= \frac{1}{\pi} \cdot \sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{\pi}\left(\sin\frac{\pi}{2} - \sin\frac{-\pi}{2}\right) \\
&= \frac{1}{\pi}\left(1 - (-1)\right) \\
&= \frac{2}{\pi}
\end{aligned}$$

Thus the average value of $f(x) = \frac{2}{\pi}$

3 Arc Length

This formula for arc length only applies to standard, Cartesian coordinate functions. It is not to be used for the arc length of a polar function, nor should it be confused for the arc length for the circle. The arc length of a function from $x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, dx$$

or alternatively:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

3.1 Derivation

The arc length can be derived utilizing calculus, algebra and the Pythagorean theorem. Recall that by the Pythagorean theorem, the hypotenuse z of a right triangle can be given by $z^2 = x^2 + y^2$. This means that in terms of calculus, $\mathrm{d}z^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. From this, we get:

$$\mathrm{d}z = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

Naturally, the entire arc length of a function is the sums of the hypotenuses formed by the tiny triangles with legs $\mathrm{d}x$ and $\mathrm{d}y$. Therefore, we can write the integral:

$$L = \int \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

We factor a $\mathrm{d}x^2$ outside:

$$\begin{aligned} L &= \int \sqrt{\mathrm{d}x^2 \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)} \\ &= \int \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, dx \end{aligned}$$

Thus we get the formula. Of course, if we have to integrate with respect to y, we can just factor our a $\mathrm{d}y^2$ instead and get

$$L = \int \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\, dy$$

4 Volume

Given a function $f(x)$, sometimes we can be asked to determine the volume of a solid created by either revolving it or taking the cross-section and a base. Out of these questions, revolutions are the most common. Generally, revolutions require using the disk, washer, or shell methods. Cross-sectional questions use a special cross-sectional volume method, which requires geometric understanding.

4.1 Disk

Given the function $f(x) = \sqrt{x}$, determine the volume of the solid generated when this function is bounded by the y-axis and the line $x = 4$, and is revolved around the x-axis. I will assume you know what this means, as the fundamental action is difficult to explain in a text-based medium.

To find the volume, we basically chop the resulting volume into a series of cylinders of infinitesimal height (disks), and then sum the volumes of all of them. As we are moving up the x-axis, this infinitesimal height will be $\mathrm{d}x$.

Since we are revolving the function around the x-axis, the radius of the disk at any given x will be the distance from $f(x)$ and the x-axis, which simplifies down to $f(x)$. Therefore, we can use the formula for volume of a cylinder to find that the volume of each disk is $V = \pi f(x)^2\, \mathrm{d}x$. Now, we can integrate from 0 to 4 to find the true volume:

$$\begin{aligned} V &= \int_0^4 \pi * f(x)^2\, dx \\ &= \pi \int_0^4 \sqrt{x}^2\, dx \\ &= \pi \int_0^4 x\, dx \\ &= \pi \cdot \frac{1}{2} x^2 \bigg|_0^4 \\ &= 8\pi \end{aligned}$$

It is of note that the general formula for a solid resulting from a function $f(x)$ bounded from $x = a$ to $x = b (a < b)$ and revolved around the x-axis is

$$V = \pi \int_a^b f(x)^2\, dx$$

4.1.1 Revolving Around Some Other Y

Sometimes, the revolution won't take place around the x-axis, and the area won't be bounded by the x-axis. Instead they will both be around some arbitrary line at $y = c$. In this case, we just modify the integral to be

$$V = \pi \int_a^b |f(x) - c|^2\, dx$$

The presence of the absolute value means we will need to determine when $f(x) > c$ and when $f(x) < c$ and integrate separately. In the case where $f(x) < c$, we swap the ordering in the absolute value and get $|c - f(x)|$.

4.2 Washer

The disk method can only deal with solid volumes, where the area is bounded by the same horizontal line we revolve around. However, sometimes our area will be instead bounded by a different line or even another function. In this case, we have to use a different formula:

$$V = \pi \int_a^b |f(x) - g(x)|^2 \, dx$$

Where $f(x)$ and $g(x)$ are the two functions.

Alternatively, if we are given that $f(x) > g(x)$ for all x, we can also use

$$V = \pi \int_a^b f(x) \, dx \; - \int_a^b g(x) \, dx$$

This second method is akin to using the disk method to find the volume of both functions and subtracting the inner volume from the entire volume.

Example: Determine the volume of a solid bounded by $x = 0$ and $x = 2$, as well as the functions $y = 5$ and $f(x) = \sqrt{(}x)$, when it is revolved around the x-axis.

In this case, from $x = 0$ to $x = 4$, $5 > \sqrt{(}x)$, so we can write the integral as

$$V = \pi \int_0^4 \left(5 - \sqrt{(}x)\right)^2 \, dx$$

And forgo the absolute value. Evaluating this, we can get

$$\begin{aligned} V &= \pi \int_0^4 \left(5 - \sqrt{(}x)\right)^2 \, dx \\ &= \pi \int_0^4 \left(25 - 10\sqrt{x} + x\right) \, dx \\ &= \pi \left(25x - \frac{20}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2\right) \Big|_0^4 \\ &= \pi \left(100 - \frac{160}{3} + 8\right) \\ &= \pi \left(108 - \frac{160}{3}\right) \end{aligned}$$

The other cases are similar. Note that if the functions intersect, and sometimes $f(x) > g(x)$, and sometimes $f(x) < g(x)$, we find their intersection points and integrate them separately, swapping the larger portion to the top, regardless of which method we use.

4.3 Shell

The shell method is commonly used when we must revolve a function around the y-axis. Sure, we could solve the equation for x and then use the disk/washer methods to find the volume, but the shell method greatly simplifies this method. Instead of dividing into disks and washers of infinitesimal height, we instead "cut" our volume into a bunch of rings, each with an infinitesimal thickness. This results in a bunch of "hollow cylinders". Since the thickness is so small, the difference in length between the inner and outer walls of the cylinder is negligible, and thus we treat it as a very thin rectangular prism, who's volume will be $L \cdot H \cdot dx$. This way, we can determine the volume.

Visualizing the shell, we can see its height is given by the actual function $f(x)$ for some x, while the "length" of our prism is the circumference of a circle with radius x, and thus $L = 2\pi x$. Knowing this, the shell method becomes:

$$2\pi \int_a^b x \cdot f(x)\, dx$$

Example:
Given the function $f(x) = x^2$ is used to form a solid by revolving the area between it and the x-axis around the y-axis, bounding it by the line $x = 4$, determine the volume of the solid.

Here, we simply plug into the formula:

$$\begin{aligned}
&2\pi \int_0^4 x \cdot x^2\, dx \\
&= 2\pi \int_0^4 x^4\, dx \\
&= 2\pi \left(\frac{1}{4}x^4\right)\Big|_0^4 \\
&= 128\pi
\end{aligned}$$

And thus find our area.

4.4 Cross-sectional Area

Cross-sectional area question are often given as: Given a function $f(x)$ bounded by a and b which defines the base of a solid, of which all it's cross sections are some shape, determine the volume of the solid.

These questions require you to know the formula for the area of whatever cross-sectional shape is given. I will simply call this formula $L(p)$, where p is some parameter that can help find the area (such as $f(x)$ itself). Common examples of shapes are equilateral triangles, squares, rectangles, or semicircles. Thus the formula for volume is given by

$$\int_a^b L(p)\, dx$$

Example:
Let $f(x) = x^2$ define the base of a solid with cross sections that are all equilateral triangles. Given that $f(x)$ is bounded by 0 and 3, determine the volume of the resulting shape.

First, we need to find what $L(p)$ is. Note that the area of an equilateral triangle is $\frac{\sqrt{3}}{4}s^2$ where s is the triangle's side length (this formula is derived from $A = \frac{1}{2}bh\sin\theta$. Thus let's let p be the side length of the triangle. Since the triangle is equilateral, the length of the base here is also the side length, and thus $p = f(x)$. Therefore:

$$L(p) = \frac{\sqrt{3}}{4}f(x)^2\, dx$$

And thus our integration is

$$\begin{aligned}
&\int_0^3 \frac{\sqrt{3}}{4}(x^2)^2\, dx \\
&= \frac{\sqrt{3}}{4} \cdot \frac{1}{5}\left(x^5\right)\Big|_0^3 \\
&= \frac{243\sqrt{3}}{20}
\end{aligned}$$

5 Polar

Polar integration is much of the same, except our function is in terms of θ and we integrate with respect to dθ. There are a couple formulae needed for polar functions. Remember an polar function can traditional x, y grid coordinates by using:

$$\begin{aligned}
x &= r(\theta)\cos\theta \\
y &= r(\theta)\sin\theta
\end{aligned}$$

5.1 Area of a Polar Function

The area of a polar function is given by the formula

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r(\theta)^2 \, d\theta$$

Example:
Find the area of the polar curve $r(\theta) = \sec\theta$ from $\frac{\pi}{3}$ to $\frac{5}{6}\pi$.

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5}{6}\pi} sec^2\theta \, d\theta \\ &= \frac{1}{2}\tan\theta \Big|_{\frac{\pi}{3}}^{\frac{5\pi}{6}} \\ &= \frac{1}{2}\left(\frac{-\sqrt{3}}{3} - \sqrt{3}\right) \\ &= \frac{2\sqrt{3}}{3} \end{aligned}$$

5.2 Arc Length of Polar Curves

The arc length of a polar curve can be given by the following formula. Note that this only applies to polar curves and should not be confused for the other arc length formulae

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r(\theta)^2 + r'(\theta)^2} \, d\theta$$

6 Parametric

The only application of parametric integration will be to find the arc length of a parametric function. It is given by the formula

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, dt$$

Simply plug in $x'(t)$ and $y'(t)$ and evaluate.

7 Differential Equations

Differential Equations make up a significant portion of the unit. A differential equation is a function expressed in terms of its derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$. To solve such equations, we treat $\mathrm{d}y$ and $\mathrm{d}x$ as separate values, thus isolating them and then integrating both sides. Following this, we then use initial values and the integration constant C to find the resulting function.

Example:
Find the particular solution to the following differential equation that passes through the point $(0, 6)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy = x$$

First, we will separate x and y from each other:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} - 2xy &= x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= x + 2xy \\ &= x(1 - 2y) \end{aligned}$$

Now, we separate dy and dx from each other, moving them to their respective sides, as well as moving the corresponding y and x terms over.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1+2y)$$
$$\frac{\mathrm{d}y}{1+2y} = x\,\mathrm{d}x$$

We then integrate both sides

$$\frac{\mathrm{d}y}{1+2y} = x\,\mathrm{d}x$$
$$\int \frac{\mathrm{d}y}{1+2y} = \int x\,\mathrm{d}x$$
$$\frac{1}{2}\ln(1+2y) = \frac{1}{2}x^2 + C$$

Note we only need to include the $+C$ term on one side of the equation.

The $\frac{1}{2}$ coefficients cancel out, so we have

$$\ln(1+2y) = x^2 + C$$
$$1+2y = e^{x^2+C}$$

Using the multiplication rule for exponents $a^m * a^n = a^{m+n}$, we can simplify our equation to

$$1+2y = e^C e^{x^2}$$

Since e^C is a constant (C is a constant), we can just write this as:

$$1+2y = Ce^{x^2}$$

Now, we can solve for C by plugging our initial value, the point $(0,6)$.

$$1+2y = Ce^{x^2}$$
$$1+2(6) = Ce^0$$
$$C = 1+2(6)$$
$$= 13$$

And with this, we can plug in 13 for C and find our final answer of

$$1+2y = 13e^{x^2}$$

There is no need for any further simplification.

7.1 The Logistic Equation

The logistic equation is a special kind of differential equation commonly used for population growth, among other things. It is characterized by the following differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{L}\right)$$

Here, P represents the size of the population, k is the growth rate of the population (which is a constant value), and L is the limiting or carrying capacity of the environment (the maximum size of the population).

To solve the logistic equation, we must employ the techniques of separating variables and partial fractions. First, we separate the variables.

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= kP\left(1-\frac{P}{L}\right) \\
\frac{\mathrm{d}P}{P\left(1-\frac{P}{L}\right)} &= k\,\mathrm{d}t \\
\frac{\mathrm{d}P}{P\left(\frac{L-P}{L}\right)} &= k\,\mathrm{d}t \\
\frac{L\,\mathrm{d}P}{P\left(L-P\right)} &= k\,\mathrm{d}t
\end{aligned}$$

Integrating both sides we get:

$$\begin{aligned}
\int \frac{L\,\mathrm{d}P}{P\left(L-P\right)} &= \int k\,\mathrm{d}t \\
L\int \frac{1}{P\left(L-P\right)}\,dP &= k\int \,dt \\
&= kt + C
\end{aligned}$$

The integration of the right side was simple enough. To integrate the left side we will invoke partial fractions:

$$\begin{aligned}
\text{Let } \frac{1}{P\left(L-P\right)} &= \frac{A}{P} + \frac{B}{L-P} \\
&= \frac{AL - AP + BP}{P\left(L-P\right)}
\end{aligned}$$

As our denominators are now equal, we get the following equation:

$$\begin{aligned}
1 &= AL - AP + BP \\
&= (B-A)P + AL
\end{aligned}$$

By inspecting coefficients, we can clearly see that

$$\begin{aligned}
AL &= 1 \\
A &= \frac{1}{L}
\end{aligned}$$

and

$$\begin{aligned}
B - A &= 0 \\
A &= B \\
B &= \frac{1}{L}
\end{aligned}$$

Plugging back, we obtain:

$$\begin{aligned}
&L\int \frac{1}{P\left(L-P\right)}\,dP \\
&= L\left(\int \frac{\frac{1}{L}}{P}\,dP + \int \frac{\frac{1}{L}}{L-P}\,dP\right) \\
&= L\left(\frac{1}{L}\int \frac{1}{P}\,dP + \frac{1}{L}\int \frac{1}{L-P}\,dP\right) \\
&= \ln|P| - \ln|L-P| \\
&= \ln\left|\frac{P}{L-P}\right|
\end{aligned}$$

Therefore our logistic equation has become:

$$\ln\left|\frac{P}{L-P}\right| = kt + C$$

Since negative populations and carrying capacities don't make sense, we can forgo the absolute value and just write

$$\begin{aligned}
\ln\left(\frac{P}{L-P}\right) &= kt + C \\
\frac{P}{L-P} &= e^{kt+C} \\
&= Ce^{kt}
\end{aligned}$$

Now, given our initial population was P_0 at time $t = 0$, we can solve for C

$$\begin{aligned}
\frac{P}{L-P} &= Ce^{kt} \\
\frac{P_0}{L-P_0} &= Ce^{0} \\
C &= \frac{P_0}{L-P_0}
\end{aligned}$$

Isolating P, we get

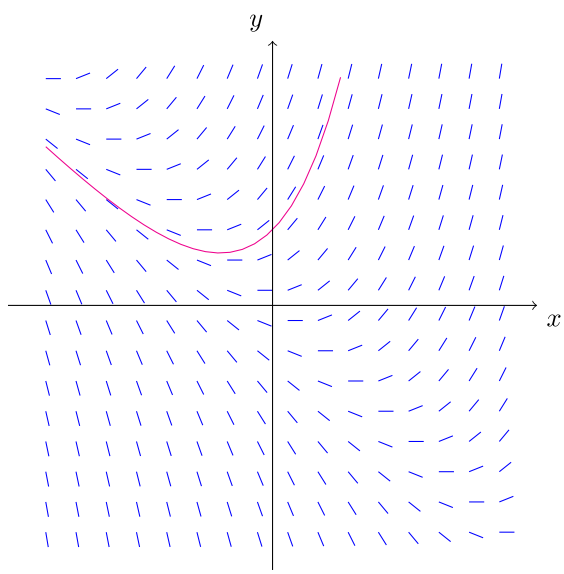
$$\begin{aligned}
\frac{P}{L-P} &= e^{kt+C} \\
P &= (L-P)\,Ce^{kt} \\
P + PCe^{kt} &= LCe^{kt} \\
P\left(1 + Ce^{kt}\right) &= LCe^{kt} \\
P &= \frac{LCe^{kt}}{1+Ce^{kt}} \\
P &= \frac{L\left(\frac{P_0}{L-P_0}\right)e^{kt}}{1+\left(\frac{P_0}{L-P_0}\right)e^{kt}} \\
P &= \frac{LP_0e^{kt}}{(L-P_0)+P_0e^{kt}}
\end{aligned}$$

And this is the solution to the logistic equation, though you may encounter different methods of writing this. On the test, you can choose to memorize this or integrate it yourself (unless the question specifically asks for you to integrate. If you memorize, note that this solution only works for the initial value $P = P_0$ at $t = 0$.

7.2 Slope Field

The slope field of a function is a graphical representation of all the slopes given by the function's differential equation for all possible integer values of (x, y). A greater slope is represented by a steeper line at that point. An example is given below:

Slope field of $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$.



In the figure above, we can see that the slope field dictates how the function (the magenta line) will curve for any given initial value. The function will always follow it's slope field.

Slope field questions come in three forms:

7.2.1 Drawing a Slope Field

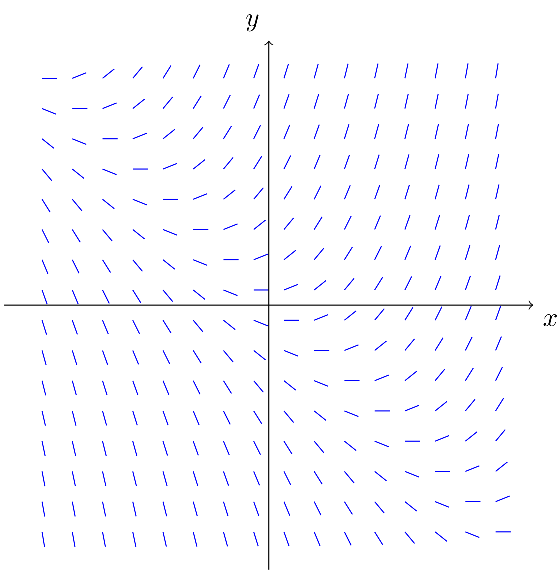
Questions that require you to draw a slope field will generally present you with a function $f(x)$ or it's differential equation $\left(\frac{\mathrm{d}y}{\mathrm{d}x} = \dots\right)$. To solve these questions, you just plug in all the possible pairs of (x, y) that you can and find the values of the slopes for all of them, and then draw the corresponding line on the graph.

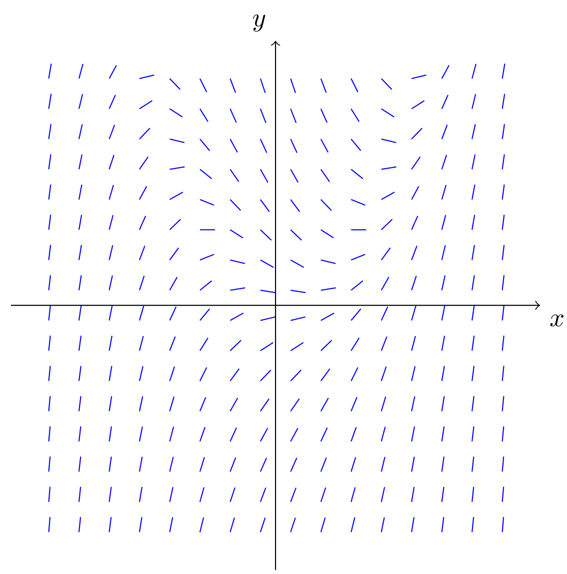
The lines you draw on the graph have to be accurate. A slope of zero should be flat, and a slope of 1 should be shallower than a slope of 2. Negative slopes should point in the right direction.

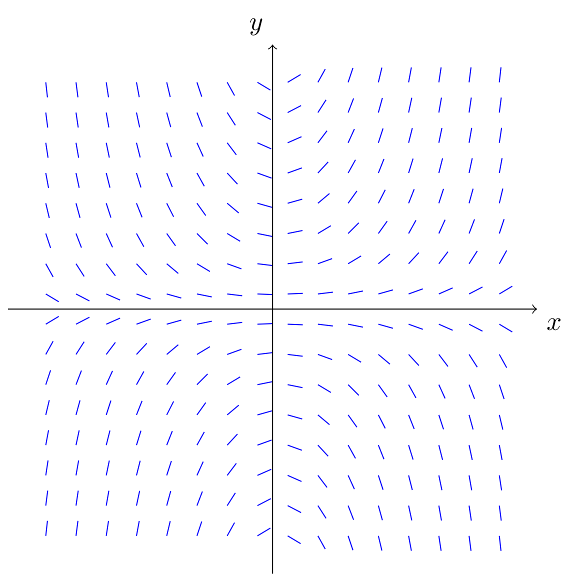
7.2.2 Matching Slope Field to Function

Sometimes, you will be given a function, and asked to determine which of the following slope fields matches the function. In this case, we can use process of elimination, plugging in random or choice values for x and y until we have found discrepancies with all of the slope fields except for one.

Example:
Which of the following slope fields matches the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - y$? Note that the grid increases per tick by $\frac{1}{3}$, so moving right three markings accounts for moving by $x = 1$.







We begin evaluating this by plugging in random points and seeing what happens. Lets try the point $(-1,-1)$. At this point, our differential equation yields $(-1)^2 - (-1) = 2$, which is a positive slope. However, looking at our first graph, it shows a negative slope at $(-1,-1)$, and so it is eliminated.

Next, we will look at the point $(1,1)$. By the equation, our slope should be $1^2 - 1 = 0$. However, graph 3 indicates a positive slope there (3 ticks up and right), which is not what we expected.

Therefore, our answer must be the second slope field.

7.2.3 Matching Functions to Slope Field

The reverse of the previous type of question asks you to map a slope field to it's corresponding function. Likewise, in this case, we once again use the process of elimination. We can choose choice values on the slope fields (generally when the slope is 0 or undefined – horizontal and vertical lines respectively), and plug them in and see which functions do not match. Then, if needed, we plug in other values and compare the signs on the slopes. In this way, we can determine the correct function for the slope field.

8 Euler's Approximation Method

Euler's approximation method can approximate the value of a function given its differential equation, and a starting point. To use this method, we chop the function into n subdivisions, then use the local linear approximation on each of the subdivisions to find the new value.

Let us define the step length Δx to be $\Delta x = \frac{b-a}{n}$, where b is the target to approximate and a is our initial reference value. Then, we repeat the following calculation until we reach b

$$Y_n = Y_{n-1} + \Delta x f'(X_{n-1}, Y_{n-1})$$

Where Y_n is the y-value of the current step, and X_{n-1}, Y_{n-1} are the x and y we found for the previous iteration of our method. f' is the differential equation we were given.

Example:
Given that the following differential equation passes through $(0,0)$, approximate the value of $f(1)$ using two subdivisions:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$$

We will use a table to keep track of our current y and x. Clearly, since we have two subdivisions, we need to repeat our calculations twice. Additionally, we can find that our $\Delta x = \frac{1-0}{2} = \frac{1}{2}$.

n	X_n	Y_{n-1}	Slope $\cdot\, \Delta x$	Y_n
0	0			0
1	0.5	0	$(0+0)\cdot 0.5 = 0$	$0+0=0$
2	1	0	$(0.5+0)\cdot 0.5 = 0.25$	$0+0.25=0.25$

Therefore our final answer is 0.25.

9 Rate In, Rate Out

Rate in, rate out problems are usually problems that give two functions that represent the rate in and rate out of a scenario (eg. rates of water flowing in and leaking out of a barrel). Such questions usually involve integrating the rate to find the volume of water (or some other metric in the question) that was added/removed, or taking the derivative of the rate to find the rate of change of the rate (akin to acceleration). We will use the following question as an example:

Water flows into a barrel containing $3\,\mathrm{L}$ at a rate $E(t) = 5e^{-0.2t}$ where t is measured in minutes and leaks out the barrel at a constant $0.6\,\mathrm{L/min}$.

9.1 Net Change

Suppose the question asked: Determine the volume of water that flowed into the barrel between $t = 0$ and $t = 5$ minutes.

Since the question only asks for water that entered the barrel, we can ignore the rate at which water leaks. Thus, the water that entered the barrel is given by the integral:

$$\int_a^b E(t)\,dt$$

Where a and b are our time bounds, so we have:

$$\begin{aligned}
&\int_0^5 E(t)\,dt \\
&= \int_0^5 5e^{-0.2t}\,dt \\
&= -25e^{-0.2t}\Big|_0^6 \\
&= -25e^{-1} + 25 \\
&= 25 - \frac{25}{e} \\
&= 25\left(1 - \frac{1}{e}\right)
\end{aligned}$$

Since this question specified units, we must return the correct units. Thus, our answer is $25\left(1 - \frac{1}{e}\right)$ L.

If the question instead asked, find the net change of the water inside the barrel between $t = 0$ and $t = 1$, we would have to consider the rate at which water left the barrel. This rate, which we will call $L(t)$, was just given as a constant value of 0.6 L/min, and so $L(t) = 0.6$.

Now, the net change of water in the barrel is equal to the volume of water added subtracted by the volume of water that left. Therefore:

$$\begin{aligned}
\text{Net Change} &= \int_0^1 E(t)\,dt \; - \int_0^1 L(t)\,dt \\
&= -25e^{-0.2t}\Big|_0^1 -0.6t\Big|_0^1 \\
&= -\frac{25}{e^{0.2}} + 25 - 0.6(1) + 0 \\
&= 25 - 0.6 - \frac{25}{e^{0.2}} \\
&= 24.4 - \frac{25}{e^{0.2}}
\end{aligned}$$

Therefore the water in the barrel changed by $24.4 - \frac{25}{e^{0.2}}$ L.

9.1.1 Alternate Expression of Net Change

Applying the integral subtraction rule to the formula for net change, we can also have the equivalent integral of

$$\text{Net Change} = \int_a^b \left(E(t) - L(t)\right)\,dt$$

9.2 Final Value

The actual quantity of the metric in a rate in, rate out problem at time t is given by the following formula:

$$\begin{aligned}
Q &= Q_0 + \int_0^t \text{Net Change}\,dt \\
&= Q_0 + \int_0^t \left(E(t) - L(t)\right)\,dt
\end{aligned}$$

Where Q_0 is the initial quantity we had.

Going back to our original question, lets consider the following problem: Determine the volume of water in the barrel after $t = 1$ minute.

Here, we were given that our initial value of water was 2 L. Therefore, our final volume V will be:

$$V = 3 + \int_0^1 (E(t) - L(t))\, dt$$

For simplicity's and time's sake, I have chosen the exact integral bounds as the net change above, and thus the integral evaluates out to $24.4 - \frac{25}{e^{0.2}}$. On an actual test, this luxury may not be afforded, meaning you would have to calculate again. As a lazy student, I do not wish to do that here though.

$$\begin{aligned} V &= 3 + 24.4 - \frac{25}{e^{0.2}} \\ &= 27.4 - \frac{25}{e^{0.2}} \end{aligned}$$

We have $27.4 - \frac{25}{e^{0.2}}$ L

9.3 Time Until

Sometimes, you may have to calculate the time until a certain condition. For example, using our volume example, let us calculate the time until the barrel is empty. To do this, let x be the time at which the barrel is empty. Using the final value formula, we have

$$V = 3 + \int_0^x (E(t) - L(t))\, dt$$

The barrel is empty, so $V = 0$. Therefore:

$$\begin{aligned} 0 &= 3 + \int_0^x \left(5e^{-0.2t} - 0.6\right) dt \\ &= 3 + \left(-25e^{-0.2t} - 0.6t\right)\Big|_0^x \\ &= 3 + \left(-25e^{-0.2x} - 0.6x\right) + 25 \\ &= 28 - 0.6x - 25e^{-0.2x} \end{aligned}$$

Solving this equation will yield us the time needed to empty the barrel. This resulting equation is far too complicated and will not be required to be solved on any non-calculator test.