Applications of Derivatives

Bill Wang

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1 Geometry

1.1 Solids

Sphere:

$$V = \frac{4}{3}\pi r^3 \qquad A = 4\pi r^2$$

Cylinder:

$$V = \pi r^2 h \qquad A = 2\pi r(r + h)$$

Cone:

$$V = \frac{1}{3}\pi r^2 h \qquad A = \pi r(r + \sqrt{h^2 + r^2})$$

Cube:

$$V = s^3 \qquad A = 6s^2$$

1.2 Shapes

Circle:

$$A = \pi r^2 \qquad P = 2\pi r$$

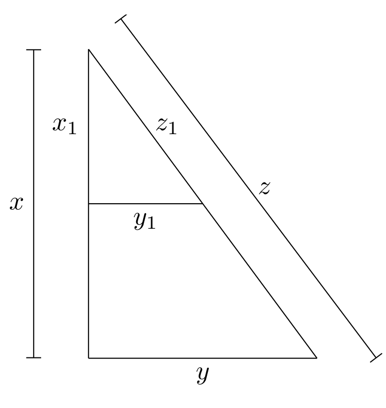
Square:

$$A = s^2 \qquad P = 4s$$

Triangle:

$$A = \frac{1}{2}bh \qquad A = \frac{1}{2}ab\sin\theta$$

1.3 Similar Triangles



Given this diagram, we have the following relationship:

$$\frac{x_1}{x} = \frac{y_1}{y} = \frac{z_1}{z}$$

Note this works for any triangle where y_1 and y are parallel. It need not be a right-angled triangle like the one shown.

2 Local Linear Approximation

To approximate the value of $f(x)$ near $x = c$, we can use the following formula:

$$f(x) \approx f(c) + f'(c)(x - c)$$

For example, given the function $f(x) = \sqrt{x}$, to approximate the value of $f(4.1)$ we can use:

$$\begin{aligned} f(x) &\approx f(c) + f'(c)(x-c) \\ f(4.1) &\approx f(4) + f'(4)(4.1-4) \\ &= \sqrt{4} + \left(\frac{1}{2\sqrt{4}}\right)(0.1) \\ &= 2 + 0.1\left(\frac{1}{4}\right) \\ &= 2.025 \end{aligned}$$

In actuality, $\sqrt{4.1} \approx 2.0248$, so our approximation was quite close.

3 Physics

3.1 Position, Velocity and Acceleration

If $x(t), v(t),$ and $a(t)$ are the position, velocity, and acceleration functions of some object with respect to time, we have the following relations:

$$\begin{aligned} v(t) &= x'(t) \\ a(t) &= v'(t) = x''(t) \end{aligned}$$

The velocity position is the derivative of the position function, and the acceleration is the derivative of the velocity function, which is also the second derivative of the position function.

An example:
The position of a particle at time t is given by the function $t^3 + 2t^2 - 6t + 12$. Determine the velocity and acceleration of the particle at $t = 3$.

$$x(t) = t^3 + 2t^2 - 6t + 12$$

$$\begin{aligned} \therefore v(t) &= x'(t) = 3t^2 + 4t - 6 \\ &\text{and} \\ a(t) &= v'(t) = 6t + 4 \end{aligned}$$

Now, we can determine $v(3)$ and $a(3)$:

$$\begin{aligned} v(3) &= 3(3)^2 + 4(3) - 6 = 33 \\ a(3) &= 6(3) + 4 = 22 \end{aligned}$$

And here we have our answer.

3.2 Direction

A particle's direction is determined by the sign of its velocity, while whether it is "speeding up" or "slowing down" is determined by the sign of the acceleration. Refer to the following table:

Sign of velocity	$+$	$+$	$-$	$-$
Sign of acceleration	$+$	$-$	$+$	$-$
Direction of motion	Right $(+)$	Right $(+)$	Left $(-)$	Left $(-)$
Change of speed	Speeding up	Slowing down	Slowing down	Speeding up

4 Related Rates

The concept of related rates is to determine the rate of change of some variable in a system given some information on how fast another variable in the system is changing. This is best illustrated with an example.

Example:
A circle's radius is increasing at a rate of 3 m/s. When the radius of the circle is 12 m, how fast is the area of the circle increasing?

To solve this, we must first realize that the area of the circle is given by the formula $A = \pi r^2$.
Taking implicit differentiation, we get:

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \pi \cdot 2r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \\ &= 2\pi r \cdot \frac{\mathrm{d}r}{\mathrm{d}t}\end{aligned}$$

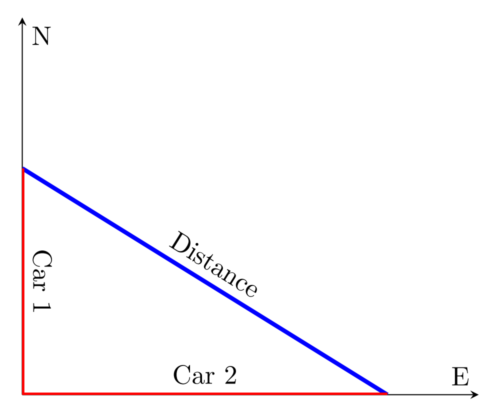
From the question, the circle's radius is increasing at 3 m/s. Therefore, $\frac{\mathrm{d}r}{\mathrm{d}t}$, which is the rate of change of r, is equal to 3. Therefore, we can plug in $r = 12$ and $\frac{\mathrm{d}r}{\mathrm{d}t} = 3$ into the formula to find $\frac{\mathrm{d}A}{\mathrm{d}t}$ at $r = 12$.

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= 2\pi \cdot 12 \cdot 3 \\ &= 72\pi\end{aligned}$$

Therefore, the area is increasing at a rate of 72π m^2/s. When solving questions of this type, note the units.

Example:
Two cars start at the origin, one facing due East, and one facing due North. At time $t = 0$, both cars simultaneously start driving. The car moving east drives at 40 km/h, where as the car driving north moves at a speed of 30 km/h. At $t = 5$ hours, how fast are the cars moving apart from each other?

Diagram:



If we let x be the distance car 1 travelled, and y be the distance car 2 has travelled, it is trivial to see that the distance (which we'll call z) is in the following relationship with x and y:

$$z^2 = x^2 + y^2$$

Taking implicit differentiation of both sides with respect to t, we get:

$$\begin{aligned}2z \cdot \frac{\mathrm{d}z}{\mathrm{d}t} &= 2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}t} \\ z \cdot \frac{\mathrm{d}z}{\mathrm{d}t} &= x \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + y \cdot \frac{\mathrm{d}y}{\mathrm{d}t}\end{aligned}$$

Once again, we know something about $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$. Since car 1 moves at 30 km/h, $\frac{\mathrm{d}x}{\mathrm{d}t} = 30$. Likewise, $\frac{\mathrm{d}y}{\mathrm{d}t} = 40$. Additionally, at $t = 5$ we can also find out the values of $x, y,$ and z. Using

$$\text{Distance} = \text{Speed} \cdot \text{Time}$$

We work out that $x = 150$ and $y = 200$ at $t = 5$. Now, we can plug in those values into the Pythagorean Theorem. Doing so, we can find that $z = 250$. Now, we can plug in all these numbers back into our implicit differentiation:

$$
\begin{aligned}
z \cdot \frac{\mathrm{d}z}{\mathrm{d}t} &= x \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + y \cdot \frac{\mathrm{d}y}{\mathrm{d}t} \\
250 \cdot \frac{\mathrm{d}z}{\mathrm{d}t} &= 150 \cdot 30 + 200 \cdot 40
\end{aligned}
$$

We now have one variable remaining: $\frac{\mathrm{d}z}{\mathrm{d}t}$. This is what we want. Solving for $\frac{\mathrm{d}z}{\mathrm{d}t}$, we get:

$$
\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{150 \cdot 30 + 200 \cdot 40}{250} \\
&= \frac{12500}{250} \\
&= 50
\end{aligned}
$$

Thus, the distance between the cars is increasing at a rate of 50 km/h at $t = 5$ hours.

5 Extreme Values

5.1 Local Minima and Maxima

A local minimum or maximum is a point on the graph of $f(x)$ where both the values to the immediate left and right of it on the function are **BOTH** less than (local maximum) or greater than (local minimum) it. To find this, we have two tests.

5.1.1 First Derivative Test

The first derivative test states that a point $x = c$ on the graph of $f(x)$ is a local extreme value (either a local minimum or maximum) if $f'(c) = 0$ and $f'(x)$ switches sign at this value. In particular, if $f'(x)$ switches signs from $-$ to $+$ at $x = c$, there is a local minimum at $x = c$. If $f'(x)$ instead switches signs from $+$ to $-$, there is a local maximum instead.

5.1.2 Second Derivative Test

The second derivative test states that at some point $x = c$, if $f(c), f'(c),$ and $f''(c)$ all exist, and $f'(c) = 0$, there may exist an extreme value at this point. Notably, if $f''(c) > 0$ at this point, there is a local minimum, and if $f''(c) < 0$, there is a local maximum. If $f''(c) = 0$, the graph is flat and simply a straight line.

5.2 Absolute Minima and Maxima

The absolute minimum of a function is some value $f(c)$ such that $f(c) \leq f(x)$ for all x in the function's domain. Likewise, the absolute maximum of a function is some value $f(c)$ such that $f(x) \leq f(c)$ for all x in the function's domain.

To find the absolute minimum or maximum, we find all the respective local extreme values of the function and find the values of both endpoints of the function. Then, we can compare all these values to see which one is the true absolute extreme value. This process is not difficult, but is time consuming.

5.3 Extreme Value Theorem

The extreme value theorem says that given a function $f(x)$ that is continuous over the closed interval $[a, b]$ (in other words continuous for all x between a and b), there will exist an absolute minimum and an absolute maximum for $f(x)$ on the open interval (a, b)

5.4 Optimization

Example:
A 10 cm by 16 cm piece of cardboard is to be made into an open-topped box by cutting identical squares off each corner and then folding it up. What is the maximum volume that can be achieved?

We can model the box with the following function:

$$
\begin{aligned}
V(x) &= (10 - 2x)(16 - 2x)(x) \\
&= 160x - 32x^2 - 20x^2 + 4x^3 \\
&= 4x^3 - 52x^2 + 160x
\end{aligned}
$$

Clearly, $0 < x < 5$, otherwise we will not create a box at all. To find the possible maximum value, we need to determine local maximum. We first find $V'(x)$.

$$\begin{aligned} V(x) &= 4x^3 - 52x^2 + 160x \\ V'(x) &= 12x^2 - 104x + 160 \end{aligned}$$

To find local extreme values, we set $V'(x) = 0$. Therefore:

$$\begin{aligned} 0 &= 12x^2 - 104x + 160 \\ &= 3x^2 - 26x + 40 \end{aligned}$$

Using the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{26 \pm \sqrt{26^2 - 4 * 3 * 40}}{6} \\ &= \frac{26 \pm 2\sqrt{169 - 3 * 40}}{6} \\ &= \frac{26 \pm 2\sqrt{49}}{6} \\ &= \frac{26 \pm 14}{6} \\ x &= 2, \frac{20}{3} \end{aligned}$$

We apply will plot a table with these x values and the sign of the first derivative.

x	$x < 2$	$x = 2$	$2 < x < \frac{20}{3}$	$\frac{20}{3}$	$x > \frac{20}{3}$
$f'(x)$	+	N/A	−	N/A	+

Here, we can see that the sign of the first derivative switches signs from positive to negative at $x = 2$. By the first derivative test, there is a local maximum here. It is also the only maximum in the domain. Therefore the maximum volume occurs at $x = 2$. The maximum volume is thus $V(2) = 144\,\text{cm}^3$.

Note that when solving these types of optimization questions, it is important to create the table shown above (though a number line is good too) and to write the little passage about the first derivative test and it being the only maximum/minimum in the domain. Otherwise marks will be lost!

6 Curve Sketching

6.1 Concavity

The concavity of a function relates to the double derivative of the function. If $f''(x) > 0$ at x, the graph is concave up. If $f''(x) < 0$ at x, the graph is said to be concave down. Concavity is useful in sketching curves.

6.2 Points of Inflection

Points of inflection occur when the concavity of the function changes signs from negative to positive (or vice versa). Note that simply going to zero is not enough, it **MUST** change signs.

To find points of inflection, we first find the roots of $f''(x)$. Then, we check each of the points to see if $f''(x)$ does indeed change signs at that point. If it does, we have a point of inflection.

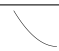
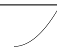
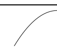
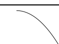
6.3 Curve Properties

When sketching a curve, the maximum detail is not required on the curve. However, the following must be represented on the graph:

- Roots of the function.
- Y-intercept of the function.
- Points of inflection.
- Local minima and maxima.
- Curve shape obeys concavity and slope.

6.4 Derivative and Concavity

Together, the derivative and concavity of a function determine how a function will curve at any given time. The following table indicates the general shape of the curve for different combinations of slope and concavity.

Derivative	$-$	$+$	$+$	$-$
Concavity	$+$	$+$	$-$	$-$
Shape				

6.5 Critical Points

A critical point c of a function occurs for any $x = c$ where $f(c)$ exists and $f'(c) = 0$. These are very useful in finding minima and maxima, as well as curve sketching.

6.6 Sketching a Curve

The process of sketching a curve can be broken down into a few steps.

Example:
Sketch the graph of $f(x) = \frac{\ln x}{x}$

First, we find the zeros of this function:

$$\begin{aligned} \frac{\ln x}{x} &= 0 \\ \ln x &= 0 \\ x &= e^0 \\ x &= 1 \end{aligned}$$

Therefore, $x = 1$ is a zero of this function.

Then, we find the y-intercept of the function. The y-intercept occurs when $x = 0$. In this case, we do not have a y-intercept.

Next, we find the asymptotes of the function. This is a limits calculation, and is too trivial to be shown here. The asymptotes are at $x = 0$ and $y = 0$ respectively.

Next, we find the critical points of the function:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{\ln x}{x} \\ &= \frac{\frac{1}{x} \cdot x - \ln x}{x^2} \\ &= \frac{1 - \ln x}{x^2} \end{aligned}$$

Setting $f'(x) = 0$ gets us:

$$\begin{aligned} 0 &= \frac{1 - \ln x}{x^2} \\ &= 1 - \ln x \\ \ln x &= 1 \\ x &= e \end{aligned}$$

Therefore, we have a critical point at $x = e$.

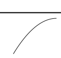
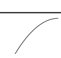
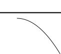
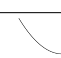
After this, we must find points where concavity changes. In other words, we find all the points of inflection on the curve. To do so, we first take the second derivative of the function:

$$\begin{aligned} f''(x) &= \frac{\mathrm{d}f'(x)}{\mathrm{d}x} \\ &= \frac{-\frac{1}{x} \cdot x^2 - 2x(1 - \ln x)}{x^4} \\ &= -\frac{1 + 2(1 - \ln x)}{x^3} \\ &= -\frac{3 - 2\ln x}{x^3} \\ &= \frac{2\ln x - 3}{x^3} \end{aligned}$$

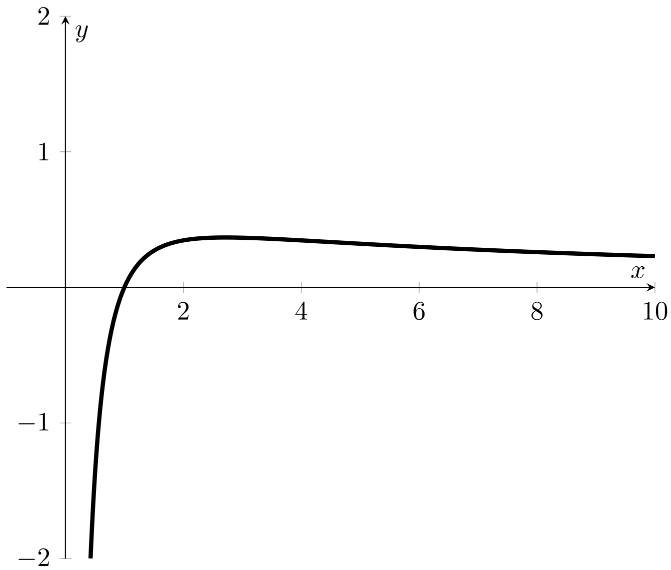
Solving for $f''(x) = 0$ gets us:

$$\begin{aligned} 0 &= \frac{2\ln x - 3}{x^3} \\ &= 2\ln x - 3 \\ \ln x &= \frac{3}{2} \\ x &= e^{3/2} \end{aligned}$$

Now, we can create the following table. We plug in all of the x values of interest:

x values	$x < 1$	1	$1 < x < e$	e	$e < x < e^{3/2}$	$e^{3/2}$	$x > e^{3/2}$
$f(x)$	$-$	0	$+$	$\frac{1}{e}$	$+$	$\frac{3/2}{e^{3/2}}$	$+$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$-$	$-$	0	$+$
Shape		—		—		—	

If we use the shapes we determined and draw our graph connecting our certain points, we should get something similar to the following graph.



I have not labelled the maximum point, intercepts, or asymptotes. These will need to be labelled on a graph. However, the graph does not need to be drawn very accurately, and a rough sketch of the curves will be fine.

Curve sketching is not difficult, but it is very time consuming and liable to calculation errors. If time permits, it may be wise to attempt curve sketching questions last.